ME 323: FLUID MECHANICS-II

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Lecture-07

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Normal Shock Wave

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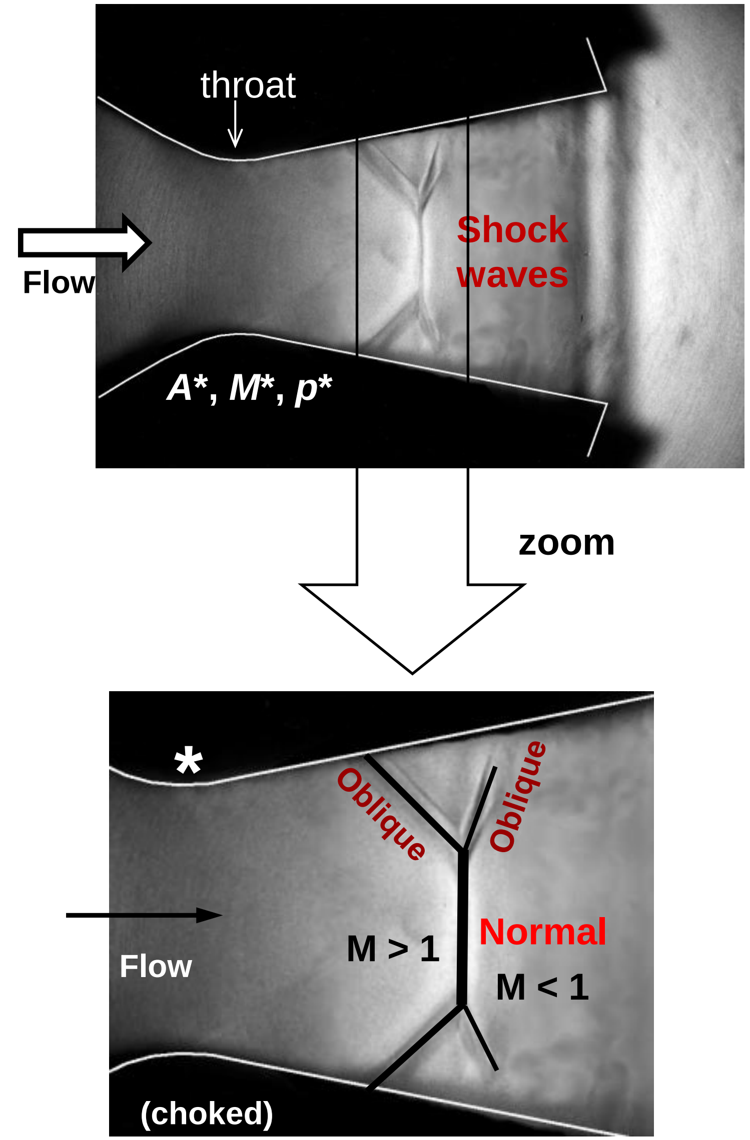
Shock Waves

Shock waves are the natural consequences of high speed flows. There are many aerodynamic applications where shock waves are the integral part of the flow field. For example, supersonic nozzle flows in overexpanded condition, exhaust of rocket engines, passage of high pressure compressors and turbines, flow around supersonic aircraft (external flows), etc.

Shock wave which is at **right angle** to the flow is **normal shock wave** and which is **inclined** to the flow direction is known as **oblique shock wave**.

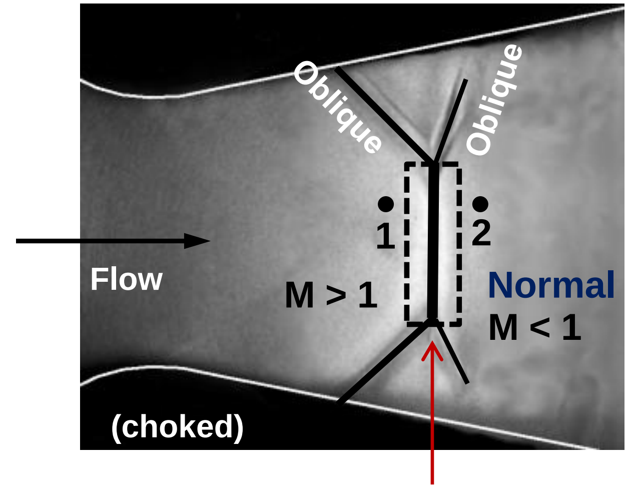
The shock wave is a very thin region and the thickness is usually on the order of a few molecular mean free paths, typically in the order of 10^{-6} m for air at standard conditions.

Since the **shock wave** is almost an instantaneous compression of the gas, it **can NOT be described using the concept of reversible isentropic process**.



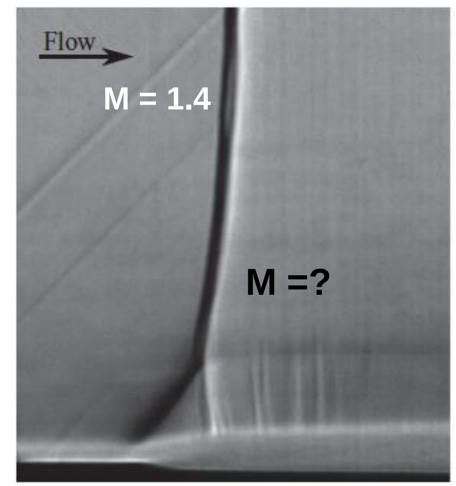
Normal Shock Waves

Though the shock wave is a **very thin** region in the flow field, there is an abrupt/instantaneous change of flow properties occur. Flow properties **just upstream of the shock wave, 1** and **just downstream of the wave, 2** vary considerably. And thus, we need to deal this tiny region separately and more carefully. The flow process from 1 to 2 is **non-isentropic.**



The flow is **supersonic (M >1) ahead** of the normal shock, and **subsonic (M <1)** behind it. (will be shown)

Furthermore, **the static pressure, temperature, and density increase across the normal shock**. However, total pressure is decreased.

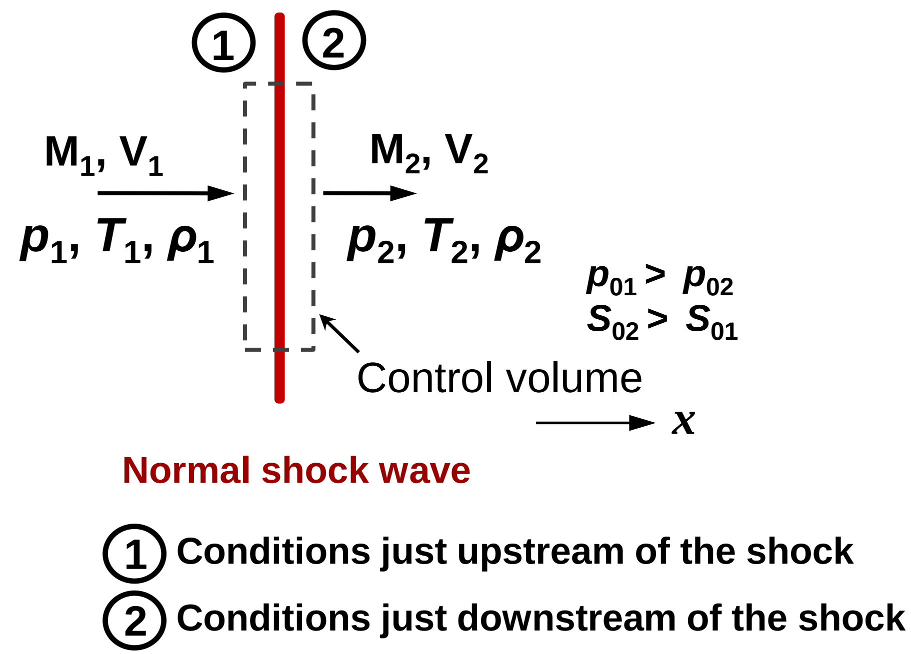


Although a shock wave can move in the flow field, we will deal with a fixed (steady) **normal shock wave in this course** (ME 323: Fluid Mechanics-II)

Normal Shock Waves

Now, we will concentrate only on the region surrounding the normal shock wave (**very thin** region). The following assumptions are considered to develop shock relations:

1. Adiabatic flow (no heat transfer to and from the CV)
2. Thickness of the shock wave is very small (**$A_1 \approx A_2$**).
3. Frictionless ideal flow ($\mu \approx 0$)
4. Steady 1-D flow.
5. Fluid behaves as ideal gas.



First, consider the **mass continuity** across the CV:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$\Rightarrow \quad \rho_1 V_1 = \rho_2 V_2 \qquad \because A_1 \approx A_2$$

$$\Rightarrow \quad \frac{p_1}{RT_1} V_1 = \frac{p_2}{RT_2} V_2 \qquad \because p = \rho RT$$

$$\Rightarrow \quad \frac{T_2}{T_1} = \frac{V_2}{V_1} \times \frac{p_2}{p_1}$$

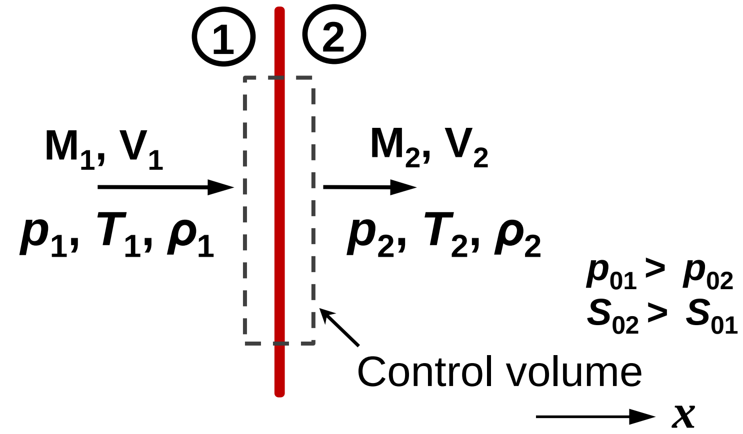
Normal Shock Waves

$$\Rightarrow \frac{T_2}{T_1} = \frac{M_2 a_2}{M_1 a_1} \times \frac{p_2}{p_1} \qquad \because \text{Mach no.}, M = \frac{V}{a}$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{M_2\sqrt{kRT_2}}{M_1\sqrt{kRT_1}} \times \frac{p_2}{p_1} \qquad \because \text{sound speed}, a = \sqrt{kRT}$$

$$\Rightarrow \sqrt{\frac{T_2}{T_1}} = \frac{M_2}{M_1} \times \frac{p_2}{p_1}$$

$$\Rightarrow \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \qquad (1)$$



Normal shock wave

① Conditions just upstream of the shock

② Conditions just downstream of the shock

Normal Shock Waves

Second, consider the **momentum equation** for the control volume:

$$\xrightarrow{+} \sum F_X = m\frac{dV}{dt}$$

$$\Rightarrow\ p_1A_1 - p_2A_2 = \dot{m}(V_2 - V_1)$$

$$\Rightarrow\ (p_1 - p_2)A_1 = \rho_2A_2V_2V_2 - \rho_1A_1V_1V_1 \qquad \because\ \rho_2A_2V_2 = \rho_1A_1V_1$$

$$\Rightarrow\ p_1 - p_2 = \rho_2{V_2}^2 - \rho_1{V_1}^2 \qquad \because\ A_1 \approx A_2$$

$$\Rightarrow\ p_1 + \rho_1{V_1}^2 = p_2 + \rho_2{V_2}^2$$

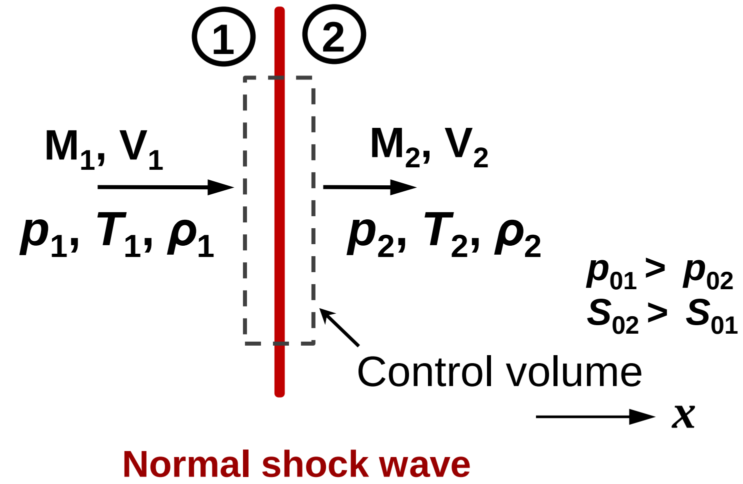
$$\Rightarrow\ p_1 + \frac{p_1}{RT_1}{V_1}^2 = p_2 + \frac{p_2}{RT_2}{V_2}^2 \qquad \because\ p = \rho RT$$

$$\Rightarrow\ p_1 + \frac{p_1k}{kRT_1}{V_1}^2 = p_2 + \frac{p_2k}{kRT_2}{V_2}^2$$

$$\Rightarrow\ p_1 + \frac{p_1k}{{a_1}^2}{V_1}^2 = p_2 + \frac{p_2k}{{a_2}^2}{V_2}^2 \qquad \because\ a = \sqrt{kRT}$$

$$\Rightarrow\ p_1 + p_1k{M_1}^2 = p_2 + p_2k{M_2}^2$$

$$\Rightarrow\ \frac{p_2}{p_1} = \frac{1+k{M_1}^2}{1+k{M_2}^2} \qquad (2)$$



① Conditions just upstream of the shock

② Conditions just downstream of the shock

Normal Shock Waves

Third, consider the **energy equation for adiabatic process and no shaft work** inside the control volume:

$$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}+\frac{V_1^{\,2}}{2}=\left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2}+\frac{V_2^{\,2}}{2}$$

$$\Rightarrow \left(\frac{k}{k-1}\right)RT_1+\frac{V_1^{\,2}}{2}=\left(\frac{k}{k-1}\right)RT_2+\frac{V_2^{\,2}}{2}$$

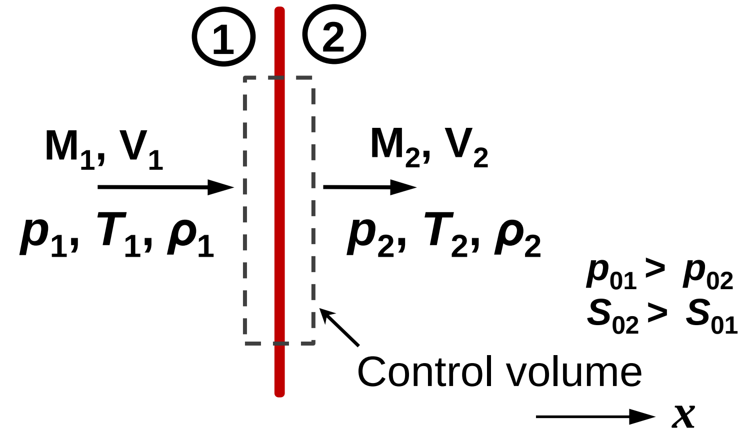
$$\Rightarrow \left(\frac{k}{k-1}\right)RT_1\left[1+\frac{k-1}{kRT_1}\frac{V_1^{\,2}}{2}\right]=\left(\frac{k}{k-1}\right)RT_2\left[1+\frac{k-1}{kRT_2}\frac{V_2^{\,2}}{2}\right]$$

$$\Rightarrow T_1\left(1+\frac{k-1}{kRT_1}\frac{V_1^{\,2}}{2}\right)=T_2\left(1+\frac{k-1}{kRT_2}\frac{V_2^{\,2}}{2}\right)$$

$$\Rightarrow T_1\left(1+\frac{k-1}{2}\frac{V_1^{\,2}}{a_1^{\,2}}\right)=T_2\left(1+\frac{k-1}{2}\frac{V_2^{\,2}}{a_2^{\,2}}\right) \qquad \because a=\sqrt{kRT}$$

$$\Rightarrow T_1\left(1+\frac{k-1}{2}M_1^{\,2}\right)=T_2\left(1+\frac{k-1}{2}M_2^{\,2}\right)$$

$$\Rightarrow \frac{T_2}{T_1}=\frac{1+\frac{k-1}{2}M_1^{\,2}}{1+\frac{k-1}{2}M_2^{\,2}} \qquad (3)$$



Normal shock wave

① Conditions just upstream of the shock

② Conditions just downstream of the shock

Normal Shock Waves

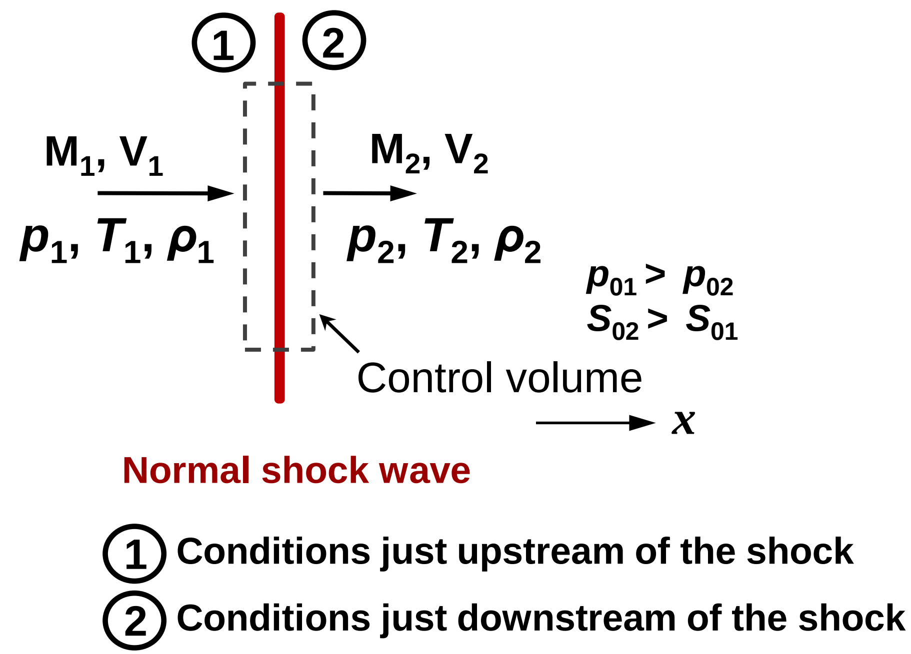
Now, use Eqn. (2) and Eqn. (3) in Eqn. (1):

$$\Rightarrow \quad \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \qquad (1)$$

$$\Rightarrow \frac{1+\frac{k-1}{2}M_1^{\ 2}}{1+\frac{k-1}{2}M_2^{\ 2}} = \left(\frac{1+kM_1^{\ 2}}{1+kM_2^{\ 2}}\right)^2 \left(\frac{M_2}{M_1}\right)^2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$



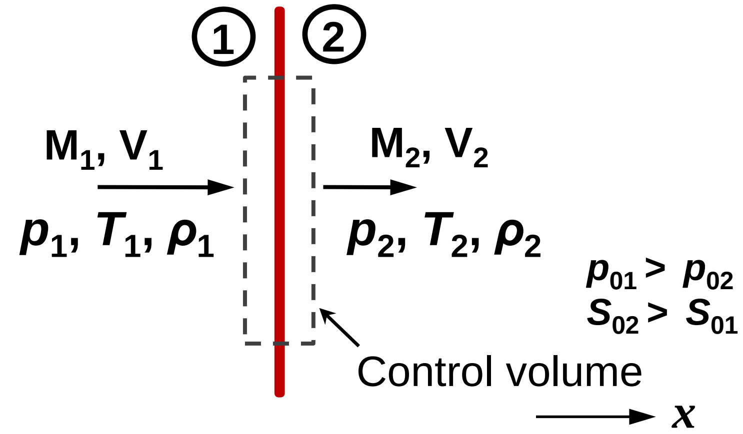
$$\Rightarrow M_2^{\ 2} = \frac{(k-1)M_1^{\ 2}+2}{2kM_1^{\ 2}-(k-1)}$$

$M_2 = f\,(M_1)$

This is very useful and important **shock relation**, **relating the downstream Mach no. M_2 to upstream Mach no. M_1 only**.

$$\Rightarrow M_2^{\ 2} = \frac{(k-1)M_1^{\ 2} + 2}{2kM_1^{\ 2} - (k-1)}$$

Downstream Mach number in terms of upstream Mach number, M_1.



Normal shock wave

(1) **Conditions just upstream of the shock**

(2) **Conditions just downstream of the shock**

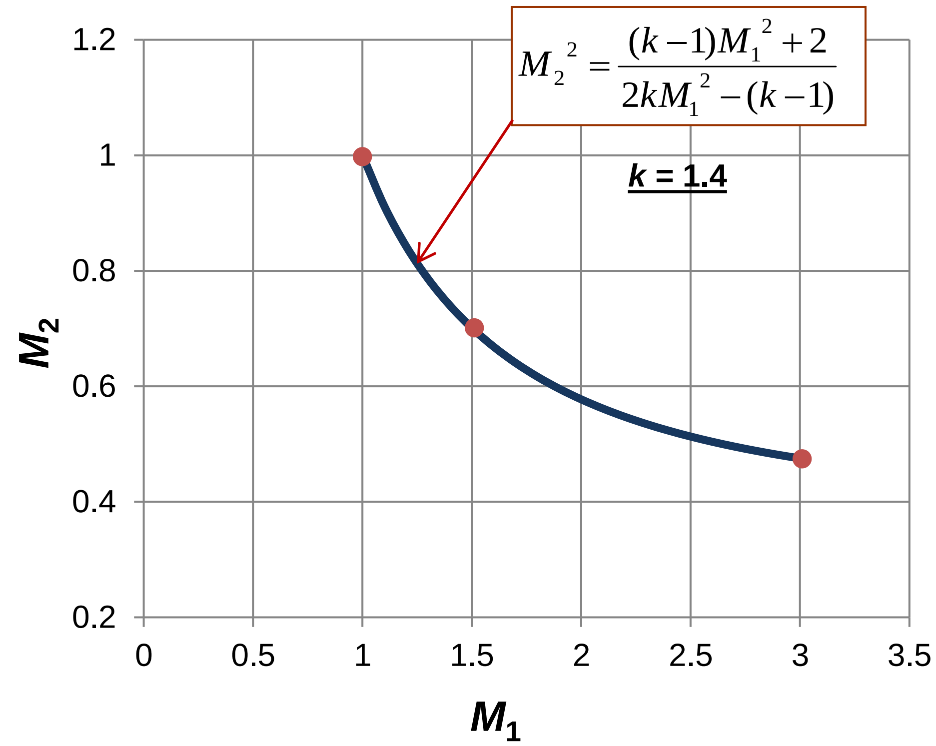
When M_1 = 1, then M_2 = 1; This is the case of an infinitely weak normal shock, which is defined as a **Mach wave.**

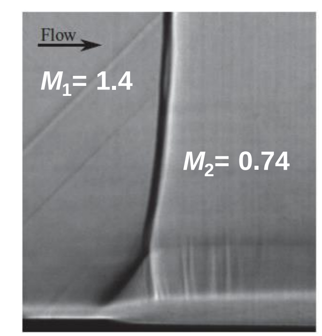
In contrast, as M_1 increases above 1, the normal shock becomes stronger and M_2 becomes progressively less than 1.

$$M_1 \rightarrow \infty : M_2 = \sqrt{\frac{k-1}{2k}}$$

Normal Shock Waves

- M_2 is always less than 1 if M_1 supersonic.
- A normal shock wave decelerates a flow almost discontinuously from supersonic (M>1) to subsonic condition (M<1).
- The static pressure downstream the shock wave increases significantly compared to upstream condition.
- This adverse pressure gradient generates some sort of drag which is called the "**wave drag**" and it has no relation with frictional behavior of fluid with this type of drag.
- In supersonic flow, **wave drag** is most crucial to deal with.
- There will be a **loss of total pressure** due to appearance of shock waves. ($p_{02} < p_{01}$)





Normal Shock Waves

The Mach no. just downstream of the shock is related to the Mach no. just upstream of the shock according to:

$$M_2^{\,2} = \frac{(k-1)M_1^{\,2} + 2}{2kM_1^{\,2} - (k-1)}$$

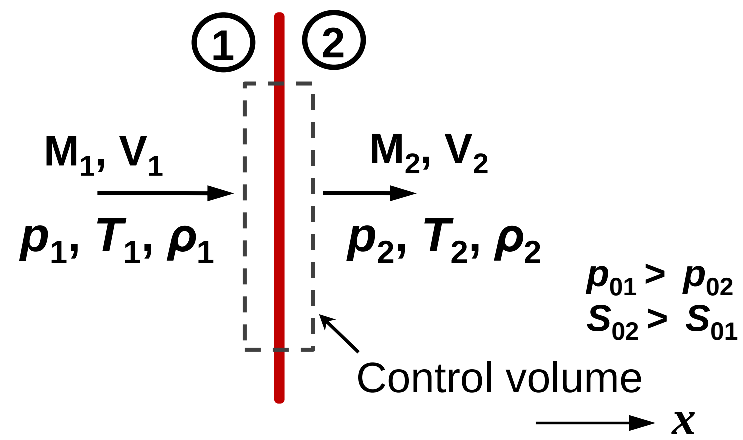
Use this expression in the relation of static pressure ratio,

$$\frac{p_2}{p_1} = \frac{1+kM_1^2}{1+kM_2^2}$$

$$= \frac{1+kM_1^2}{1+k\,\dfrac{(k-1)M_1^{\,2}+2}{2kM_1^{\,2}-(k-1)}}$$

$$= \frac{(1+kM_1^2)(2kM_1^{\,2}-(k-1))}{2kM_1^{\,2}-(k-1)+k(k-1)M_1^{\,2}+2k}$$

$$= \frac{(1+kM_1^2)(2kM_1^{\,2}-(k-1))}{kM_1^{\,2}(2+k-1)-(k-1)+2k}$$



Normal shock wave

(1) Conditions just upstream of the shock

(2) Conditions just downstream of the shock

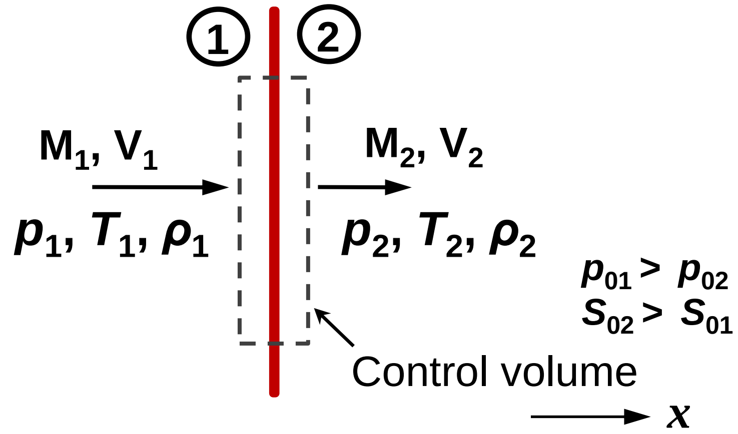
$$\Rightarrow \frac{p_2}{p_1} = \frac{(1+kM_1^2)(2kM_1^2-(k-1))}{(k+1)(1+kM_1^2)}$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{2kM_1^2-(k-1)}{(k+1)}$$

$$\boxed{\Rightarrow \frac{p_2}{p_1} = \frac{2kM_1^2}{k+1} - \frac{k-1}{k+1}}$$

$$\frac{p_2}{p_1} = f(M_1)$$

Relation of pressure ratio in terms of upstream Mach number, M_1.



Normal shock wave

① Conditions just upstream of the shock

② Conditions just downstream of the shock

In case of air flows (k = 1.4);

$$\boxed{\frac{p_2}{p_1} = \frac{1}{6}\left(7M_1^2 - 1\right)}$$

Normal Shock Waves

Again the **shock strength** is defined as,

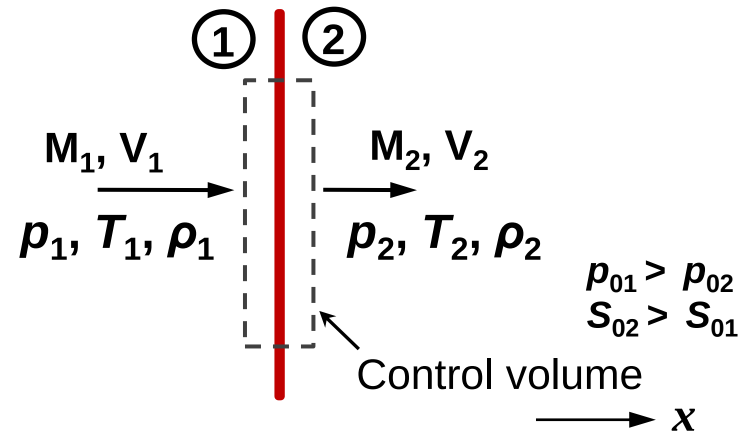
$$\beta = \frac{\Delta p}{p_1} = \frac{p_2 - p_1}{p_1}$$

$$\Rightarrow \beta = \frac{p_2}{p_1} - 1$$

$$\Rightarrow \beta = \left\{ \frac{2kM_1^{\ 2}}{k+1} - \frac{k-1}{k+1} \right\} - 1$$

$$\Rightarrow \beta = \frac{2k}{k+1}(M_1^{\ 2} - 1)$$

Shock strength in terms of upstream Mach number, M_1.



Normal shock wave

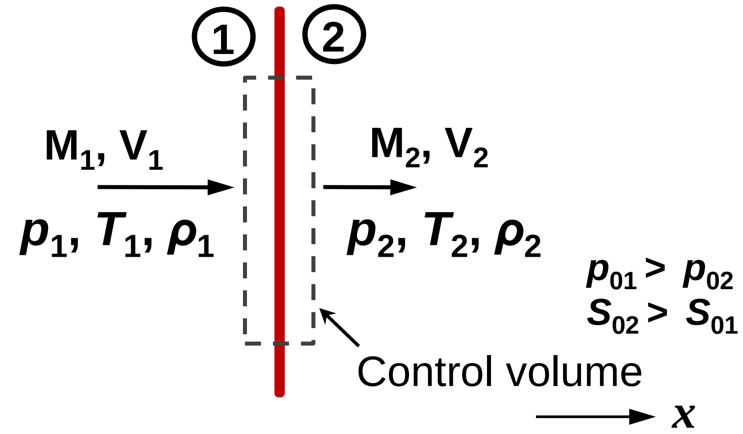
① Conditions just upstream of the shock

② Conditions just downstream of the shock

Normal Shock Waves

Now the **change of entropy** across the shock is;

$$\frac{s_2 - s_1}{R} = \frac{c_p}{R}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1} \qquad \text{(Thermodynamic relation)}$$



Normal shock wave

(1) Conditions just upstream of the shock

(2) Conditions just downstream of the shock

Recall the isentropic relations for compressible flows:

$$T_0 = T\left(1 + \frac{k-1}{2}M^2\right)$$

$$p_0 = p\left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$$

$$\therefore \frac{T_2}{T_1} = \frac{\dfrac{T_{02}}{\left(1+\dfrac{k-1}{2}M_2^{\,2}\right)}}{\dfrac{T_{01}}{\left(1+\dfrac{k-1}{2}M_1^{\,2}\right)}} = \frac{T_{02}}{T_{01}}\frac{\left(1+\dfrac{k-1}{2}M_1^{\,2}\right)}{\left(1+\dfrac{k-1}{2}M_2^{\,2}\right)} = \frac{1+\dfrac{k-1}{2}M_1^{\,2}}{1+\dfrac{k-1}{2}M_2^{\,2}}$$

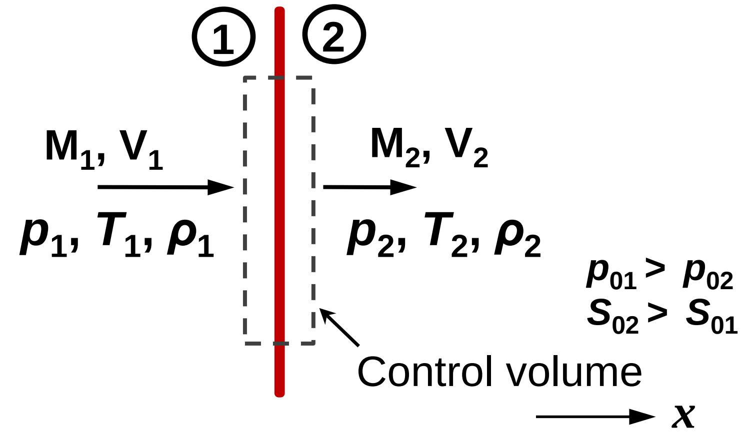
Since adiabatic process: $T_{02} = T_{01} = T_0$
Total temperature is remained same.

Normal Shock Waves

$$\therefore \frac{p_2}{p_1} = \frac{\dfrac{p_{02}}{\left(1+\dfrac{k-1}{2}M_2^{\ 2}\right)^{\frac{k}{k-1}}}}{\dfrac{p_{01}}{\left(1+\dfrac{k-1}{2}M_1^{\ 2}\right)^{\frac{k}{k-1}}}}$$

$$= \frac{p_{02}}{p_{01}}\left(\frac{1+\dfrac{k-1}{2}M_1^{\ 2}}{1+\dfrac{k-1}{2}M_2^{\ 2}}\right)^{\frac{k}{k-1}}$$

$$p_{02} \neq p_{01}$$



Normal shock wave

1 Conditions just upstream of the shock

2 Conditions just downstream of the shock

Normal Shock Waves

Now, use the expressions of temperature ratio and pressure ratio, and recall;

$$\frac{c_p}{R} = \frac{k}{k-1}$$

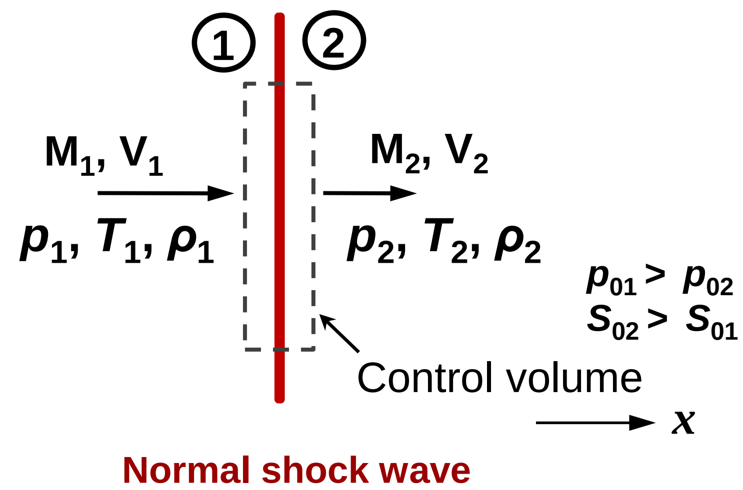
$$M_2^{\ 2} = \frac{(k-1)M_1^{\ 2} + 2}{2kM_1^{\ 2} - (k-1)}$$

The change of entropy becomes:

$$\frac{s_2 - s_1}{R} = \frac{c_p}{R}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1}$$

$$\Rightarrow \frac{s_2 - s_1}{R} = -\ln\frac{p_{02}}{p_{01}}$$

Home work
for complete derivation of this expression.



① **Conditions just upstream of the shock**

② **Conditions just downstream of the shock**

$$\frac{p_{02}}{p_{01}} = e^{-\left(\frac{s_2 - s_1}{R}\right)}$$

But from the second law of thermodynamics, $s_2 > s_1$, so that

$$p_{02} < p_{01}$$

Total pressure (pressure energy) is lost across a normal shock wave to mitigate the wave drag.